5.3) The homogeneous global Cauchy pb:

Let us consider the pb

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & x \in \mathbb{R}, \\ u_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

We notice that here, in contrast to the case of the heat eq., we have to assign two initial conditions: the initial position g and the initial velocity h.

The above pb was solved by d'Alembert in 1746. The idea is the following: the equation can be written as follows:

$$u_{tt} - c^2 u_{xx} = (\partial_{tt} - c^2 \partial_{xx}) u,$$

and the operator can be factorized as follows:

$$\partial_{tt} - c^2 \partial_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x).$$

So: [set $c>0$]

$$(\partial_t - c\,\partial_x)(u_t + c u_x) = 0.$$

Let us define $v := u_t + c u_x$. Then v satisfies:

$$(\partial_t - c\partial_x) v = 0.$$

This is a transport eq. that we can solve:

$$v(x,t) = \psi(x+ct),$$

for some ψ.

Thus, from the definition of v, we get:

$$u_t - c u_x = \psi(x+ct).$$

So:

$$u(x,t) = g(x-ct) + \int_0^t \psi(x + (s-t)c + cs)\, ds$$

$$= g(x-ct) + \int_0^t \psi(x - ct + 2cs)\, ds$$

$$= g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy.$$

$x + 2cs - ct = y$

$\Downarrow$

$s = \frac{1}{2c}[y - x + ct]$

Now, by imposing the initial condition $u_t(0,x) = h(x)$, we get:

$$h(x) = -cg'(x) + \frac{1}{2c}[\psi(x)\,2c]$$

$$= \psi(x) - cg'(x) \implies \psi(x) = h(x) + cg'(x)$$

Thus:

$$\boxed{u(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy}$$

The above formula is known as <u>d'Alembert formula</u>, found by the French matematician in 1747.